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Non-Abelian harmonic oscillators and chiral theories

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Abstract. We show that a large class of physical theories which has been under intensive investigation recently, share the same geometric features in their Hamiltonian formulation. These dynamical systems range from harmonic oscillations to zw-like models and to the KdV dynamics on $\text{Diff}_0 S^1$. To the same class belong also the Hamiltonian systems on groups of maps.

The common feature of these models are the 'chiral' equations of motion allowing for so-called chiral decomposition of the phase space.

1. Introduction

It is a common impression that many of the geometrical models in field theory share some common and universal properties, despite the technical complexity of descriptions of the models. For each of these models a specific 'machinery' has been developed, making the common features less visible. Perhaps the most prominent example of this class is the Wess–Zumino–Witten field theory [15]. Both classical Hamiltonian formulation [11] and corresponding quantum theory [5] are being viewed on the grounds of the results which are known or are expected to be obtained. This indicates that in spite of the great developments achieved recently [3, 4, 6], some fundamental background is missing.

It is our hope to shed some new light on those questions by showing a large family of the models, ranging from harmonic oscillations and free motions to the dynamical systems on the groups of maps or groups of diffeomorphisms, and making the common features evident. In fact, all those models turn out to be straightforward generalizations of harmonic oscillations and free motion.

Better understanding of the geometrical nature of those models may be very helpful in quantization, as for instance it allows one to use the data of the representation theory in a more conscious (and efficient) way.

The starting point for the definition of this class of models is the group structure on the corresponding configuration space. The next ingredient is the non-canonical lifting of the left and right actions of the group on itself to the co-tangent bundle (the phase space). We equip the phase space with a symplectic form which is invariant under the above (lifted) actions and allows for their Hamiltonian realization (by momentum mappings).

The Hamiltonian defining the dynamics is just a quadratic function of these momentum mappings, and this guarantees that the equations of motion are 'chiral'.

The paper is organized as follows. The first section contains a group-theoretical approach to the case of standard harmonic oscillators and free motions. Its power consists of its

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straightforward generalizations to the class of models describing motions on group manifolds; the Hamiltonian description of these theories is contained in the next section.

The third section is devoted to presentation of some more important and more complicated examples (like the WZW model and the dynamical model on $\text{Diff}_0 S^1$ —which leads to chiral KdV equations for the momentum mappings).

In the summary we present briefly the geometrical nature of the chiral splitting.

2. The harmonic oscillation and free motions

In accordance with our promise made in the introduction we shall briefly reformulate the theory of harmonic oscillators.

Let us consider the n -dimensional real space as the configuration space of the model. This space has a structure of an n -dimensional Abelian (additive) group.

The left and right actions of this group on itself can be lifted to the action on the phase space, but this lifting is not unique. We shall consider the following actions:

$$\Phi_a^l(x, p) := (x + a, p + La) \quad (1)$$

$$\Phi_a^r(x, p) := (x - a, p + Ra) \quad (2)$$

where (x, p) describes the position and momentum and a is a (vector) parameter of translation. L and R are linear operators from the group to its dual. The above actions are nothing but affine extensions of left and right translations by means of the group co-cycles $\theta^{l(r)}(a) := L(R)a$.

By hand (for the moment) we shall equip the phase space with the following (non-canonical) symplectic structure:

$$\Omega := \langle dp, \wedge dx \rangle + \frac{1}{2} \langle (R^A - L^A) dx, \wedge dx \rangle \quad (3)$$

where \langle , \rangle stands for the pairing and R^A and L^A stand for antisymmetric parts of R and L with respect to the pairing. The symbol dx should be understood as a vector-valued 1-form (n -bein).

It is easy to check that canonical variables have the following Poisson brackets:

$$\{p_i, x^j\} = \delta_i^j \quad (4)$$

$$\{p_i, p_j\} = (R^A - L^A)_{i,j} \quad (5)$$

$$\{x^i, x^j\} = 0. \quad (6)$$

It is also easy to check that the actions (1) and (2) are Hamiltonian and admit the following momentum mappings [7]:

$$J^r(x, p) = p + (R^S - L^A)x \quad (7)$$

$$J^l(x, p) = -p + (L^S - R^A)x \quad (8)$$

where the letters with superscript S denote the symmetric parts of corresponding operators. To avoid confusion of the momentum mappings with the canonical momenta we shall call the former (abusing the terminology slightly) the chiral momenta.

Using (4) one can find that

$$\{J_i^r, J_j^r\} = (R^A + L^A)_{ij} \quad (9)$$

$$\{J_i^l, J_j^l\} = -(R^A + L^A)_{ij} \quad (10)$$

$$\{J_i^r, J_j^l\} = -(R^S + L^S)_{ij}. \quad (11)$$

Let us notice that in spite of commutativity of the Lie algebra of translations its Hamiltonian realization gets centrally extended. Such a realization is sometimes called weakly Hamiltonian [10]. In the next section we shall assume that the symmetric parts of the co-cycles vanish, thus making the left and right currents decoupled.

The Hamiltonian of the theory is simply the sum of the squares of chiral momenta:

$$\mathcal{H} := \frac{1}{4}[(J^l, J^l) + (J^r, J^r)] \tag{12}$$

where $(,)$ stands for the Euclidean scalar product. A straightforward calculation shows that

$$\frac{d^2}{dt^2}x = -\frac{1}{4}(R^t + L^t)(R + L)x \tag{13}$$

where t stands for transposition with respect to $(,)$. Let us notice that the matrix on the RHS of (13) is never positive (as we assumed that $(,)$ is Euclidean). This means that the second-order equation describes either oscillating (positive eigenvectors) or free (null vectors) motions.

It is easy to verify that the equations of motion for J^l and J^r generated by (12) are

$$\frac{d}{dt}J^l = \frac{1}{2}(L + R)\hat{J}^l \tag{14}$$

$$\frac{d}{dt}J^r = -\frac{1}{2}(L + R)\hat{J}^r \tag{15}$$

where $\hat{J}^{l,r}$ is the vector dual to $J^{l,r}$ via $(,)$. As we shall see later, they are in precise analogy with the equations of motion for chiral currents in the WZW model [6]. This is the very reason why we call J^r, J^l the chiral momenta.

3. The general case

We can now proceed to a much more general case: let us assume that the configuration space is an arbitrary group manifold G (in particular infinite-dimensional). Moreover, we shall assume that the objects we are intended to consider do exist (even in the infinite-dimensional case). This is automatically satisfied in the cases of groups of maps from compact manifolds (with the loop groups as a special case corresponding to the WZW theory).

In order to parametrize the cotangent bundle we shall trivialize the bundle by means of the left action of the group.

The points of the phase space (T^*G) are thus described by pairs (g, p) where $g \in G$ and $p \in \mathcal{G}^*$. The left and right action of the group on itself can be lifted to the action on the phase space in the following way:

$$\Phi^l_{g_0}(g, p) := (g_0g, p + \text{Ad}^*_{g_0} \theta^l(g_0)) \tag{16}$$

$$\Phi^r_{g_0}(g, p) := (gg_0^{-1}, \text{Ad}^*_{g_0} p + \theta^r(g_0)) \tag{17}$$

where θ^l and θ^r are arbitrary \mathcal{G}^* -valued group cocycles, i.e. they do satisfy

$$\theta(g_1g_2) = \text{Ad}^*_{g_1} \theta(g_2) + \theta(g_2) \tag{18}$$

and Ad^* is the co-adjoint action.

The formulae (16) and (17) are in exact correspondence to those of (1),(2). They do not look symmetrical because we are using the left trivialization. If we used the right trivialization instead, the non-locality would appear in the expression for Φ^r .

The canonical symplectic structure of the cotangent bundle is given by the differential of the Liouville form:

$$\Omega := d\alpha \quad \text{where} \quad \alpha := \langle p, g^{-1}dg \rangle \quad (19)$$

and obviously the pairing is evaluated in \mathcal{G} —the space of values of the canonical left-invariant form $g^{-1}dg$.

It is not difficult to check that neither the Liouville form nor its differential are invariant under (16), (17). This means that the actions (16), (17) cannot be realized in a Hamiltonian way.

In order to obtain an invariant symplectic form one has to add to Ω an additional term

$$\Omega_{\text{inv}} := \Omega - \langle \Sigma^l(dg g^{-1}), \wedge dg g^{-1} \rangle + \langle \Sigma^r(g^{-1}dg), \wedge g^{-1}dg \rangle \quad (20)$$

where by $\Sigma^{l,r}$ we understand the derivative of the co-cycles $\theta^{l,r}$ at the group unity:

$$\Sigma^{l,r}(X) := \lim_{t \rightarrow 0} \frac{d}{dt} \theta^{l,r}(e^{tX}) \quad \forall X \in \mathcal{G}. \quad (21)$$

By definition they are linear operators on \mathcal{G} . At this point we make an assumption that $\Sigma^{l,r}$ are $\langle \cdot, \cdot \rangle$ -antisymmetric. In the literature [13] the co-cycles with antisymmetric derivatives are called symplectic.

Let us notice that we are free to add to (20) an arbitrary closed and both left- and right-invariant 2-form on G . On a semi-simple group, however, (contrary to the Abelian case considered in section 2) the only one such form is 0 [2].

The actions (16), (17) are Hamiltonian with respect to (20) and admit the following (weak) momentum mappings:

$$J^l(g, p) := -\text{Ad}_g^* p - \theta^l(g) \quad (22)$$

$$J^r(g, p) := p - \text{Ad}_{g^{-1}}^* \theta^r(g). \quad (23)$$

The Poisson algebra of (22) and (23) with respect to Ω_{inv} has the following form:

$$\{J^l(X), J^l(Y)\} = J^l([X, Y]) + C^+(X, Y) \quad (24)$$

$$\{J^r(X), J^r(Y)\} = J^r([X, Y]) - C^+(X, Y) \quad (25)$$

$$\{J^l(X), J^r(Y)\} = 0 \quad (26)$$

where

$$C^\pm(X, Y) := C^l(X, Y) \pm C^r(X, Y) \quad (27)$$

$$C^{l,r}(X, Y) := \langle \Sigma^{l,r}(X), Y \rangle \quad (28)$$

and $\Sigma^{l,r}$ is a derivative of a respective co-cycle as in (20).

Notice that only the sum of the cocycles matters for the central extensions of the Lie algebras of left and right translations. Let us comment briefly. If one starts from a Lagrangian formulation of a theory, like in the WZW models, the canonical momentum is given by the (dual of) velocity $g^{-1}\dot{g}$. This is equivalent to choosing a trivialization in which $\theta^l = \theta^r$ ($C^- = 0$). However, on the grounds of canonical analysis C^- can take an arbitrary value, depending on the way one trivializes the cotangent bundle (i.e. on coordinates).

In order to induce the dynamics let us introduce the following quadratic Hamiltonian:

$$\mathcal{H} := \frac{1}{4}(K(J^l, J^l) + K(J^r, J^r)) \quad (29)$$

where K is some quadratic form on \mathcal{G}^* . If we assume that K is Ad^* -invariant, then (29) together with (20) give the following equations of motion:

$$\frac{d}{dt} J^l = \frac{1}{2} \Sigma^+(J^l) \quad (30)$$

$$\frac{d}{dt} J^r = -\frac{1}{2} \hat{\Sigma}^+(\hat{J}^r). \tag{31}$$

where

$$\Sigma^\pm = \Sigma^l \pm \Sigma^r \tag{32}$$

and $\hat{J}^{l,r}$ is the K -dual of $J^{l,r}$.

These equations we shall call chiral, because in the case of G being a loop group they are precisely the equations of motion for the chiral currents in the WZW theory (see below).

The equations of motion for a group point are simple:

$$\frac{d}{dt} g = \frac{1}{2} (g \hat{J}^r - \hat{J}^l g) \tag{33}$$

whereas for the canonical momenta one has:

$$\frac{d}{dt} p = -\frac{1}{2} (\text{ad}_{\hat{J}^r}^* p + \Sigma^r(\hat{J}^r) + \text{Ad}_{g^{-1}}^* \Sigma^l(\hat{J}^l)) \tag{34}$$

$$= -\frac{1}{2} (\Sigma^r(\theta^l(g^{-1})) + \Sigma^l(\theta^r(g^{-1})) - \Sigma^-(p) + [\theta^l(g^{-1}), \theta^r(g^{-1}) - 2p]). \tag{35}$$

The equations (33), (35) seem to be difficult to solve explicitly and in any case are much more complicated than (30), (31). On the other hand one can observe that the Hamiltonian is in fact a collective one in the sense of [7], i.e. it is a pull-back of a function on $\mathcal{G}^* \times \mathcal{G}^*$ by the chiral momenta (22), (23). This fact has profound geometric consequences and we shall shed some light upon it in one of the following sections.

4. Examples

To illustrate the theory let us describe briefly some examples.

4.1. The case of an finite-dimensional Lie group

As the first instance one can consider a particle moving on a semi-simple Lie group manifold. In this case all co-cycles are trivial [2], i.e. of the form

$$\theta^{l,r}(g) = \text{Ad}_g^* \mu^{l,r} - \mu^{l,r} \quad \mu^{l,r} \in \mathcal{G}^* \tag{36}$$

and then

$$\Sigma^{l,r} = \text{ad}_{g^{-1}dg}^* \mu^{l,r}. \tag{37}$$

The invariant symplectic form is simply

$$\Omega_{\text{inv}} = d\alpha + \langle \mu^l, dg g^{-1} \wedge dg g^{-1} \rangle - \langle \mu^r, g^{-1} dg \wedge g^{-1} dg \rangle. \tag{38}$$

The equations of motion generated by the Hamiltonian of (29) are analogous to the equations of a particle moving in a magnetic-like field of strength μ^+ . The equations of motion (33), (35) can be solved explicitly on any semi-simple Lie group [18], and by ‘explicitly’ we do not mean the time-ordered integrals, but rather analytical expressions depending on time and initial data (position and momentum). The trick in solving the equations consists in the proper use of chiral factorization (see below). In particular, for $\mu^+ = 0$ they describe the free motion (along the big circles for the compact G) as in [9].

4.2. Loop groups and WZW

Another example is given by the configuration space being the loop group $\mathcal{L}G$. In this case, since the group is infinite-dimensional, we should from the very beginning extract from $\mathcal{L}G^*$ the smooth part [12] by identifying it with $\mathcal{L}G$ via the non-degenerate form on the Lie algebra $\mathcal{L}G$, defined as follows:

$$K(X, Y) = \frac{1}{2\pi} \int_0^{2\pi} \mathcal{K}(X(\sigma), Y(\sigma))d\sigma \tag{39}$$

where \mathcal{K} is an Ad-invariant form on \mathcal{G} . In this case we will take non-trivial co-cycles on $\mathcal{L}G$:

$$\theta^{l,r}(g) = k^{l,r} \left(\frac{d}{d\sigma} g \right) g^{-1} k^{l,r} \in \mathbf{R} \tag{40}$$

and then

$$\Sigma^{l,r} = k^{l,r} \frac{d}{d\sigma} . \tag{41}$$

The chiral currents do satisfy the following Poisson comutation relations:

$$\{J^l(X), J^l(Y)\} = J^l([X, Y]) - (k^l + k^r) \int_0^{2\pi} \mathcal{K}(X(\sigma), \frac{d}{d\sigma} Y(\sigma))d\sigma \tag{42}$$

$$\{J^r(X), J^r(Y)\} = J^r([X, Y]) + (k^l + k^r) \int_0^{2\pi} \mathcal{K}(X(\sigma), \frac{d}{d\sigma} Y(\sigma))d\sigma \tag{43}$$

which are easily recognized as the affine algebras with underlying Lie algebra \mathcal{G} .

By introducing the chiral derivatives

$$\partial_{\pm} := \frac{\partial}{\partial t} \pm \frac{1}{2}(k^l + k^r) \frac{\partial}{\partial \sigma} \tag{44}$$

we can write the chiral equations of motion as

$$\partial_+ J^l = 0 = \partial_- J^r. \tag{45}$$

The chiral derivatives (44) one can get in a standard form by appropriate redefinition of the time variable or equivalent rescaling of the Hamiltonian.

Similarly one could introduce the chiral dynamics on an arbitrary group of maps $\mathcal{M}G$, but this time the co-cycle would not be invariant under the group of diffeomorphisms of M . In fact all the co-cycles are defined by the 1-cycles on M in this case. Such models would be of considerable physical importance as they could describe the motions in the presence of monopole-like singularities in M .

4.3. $\text{Diff}_0 S^1$ and KdV

As another example let us consider $G = \text{Diff}_0 S^1$ —the group of orientation-preserving diffeomorphisms of S^1 . The tangent space at identity is then the space of vector fields on S^1 [16]:

$$V = T_{\text{id}}G \ni \xi = \xi(\sigma) \frac{d}{d\sigma} \quad \text{where } \sigma \text{ is a coordinate on } S^1 \tag{46}$$

and the space of smooth moments can be identified with the space of quadratic differentials

$$V^* = T_{\text{id}}^*G_{\text{smooth}} \ni p = p(\sigma)d\sigma \otimes d\sigma . \tag{47}$$

The pairing is defined as contraction, and then integrating over S^1 ,

$$\langle p, \xi \rangle = \int_0^{2\pi} p(\sigma)\xi(\sigma)d\sigma \tag{48}$$

and is clearly invariant under reparametrizations of S^1 .

The main new feature is, however, the lack of a both left- and right-invariant form on V . This means that one has to be very careful about whether one is working in V or in V^* , contrary to the case of a Lie group, where the identification between \mathcal{G} and \mathcal{G}^* defined by K allows for some carelessness in this respect.

Out of many non-invariant quadratic forms on V^* let us choose the simplest one:

$$K(p, q) = \int_0^{2\pi} p(\sigma)q(\sigma)d\sigma \quad \forall p, q \in V^* . \tag{49}$$

K can be equivalently considered to be a mapping

$$K : V^* \ni p \mapsto \hat{p} = p(\sigma)\frac{\partial}{\partial\sigma} \in V . \tag{50}$$

The Hamiltonian defined as in (29) is not Ad-invariant, as it is defined by a non-invariant form (49).

Now we have to choose θ^l and θ^r . The non-trivial co-cycle is given by [19]:

$$\theta(\phi) = \text{Ad}_\phi^* S(\phi) \quad \phi \in \text{Diff}_0 S^1 \tag{51}$$

where S is the Schwarzian:

$$S(\phi) = \frac{\phi'\phi''' - \frac{3}{2}(\phi'')^2}{(\phi')^2} d\sigma \otimes d\sigma \tag{52}$$

and the primes denote differentiation. As in general this is the sum $\theta^r + \theta^l$ that is relevant for the dynamics, let us take $\theta^{r+l} \equiv \theta$.

The derivative of θ is

$$\Sigma(\eta) = \eta(\sigma)''' d\sigma \otimes d\sigma \quad \forall \eta \in V \tag{53}$$

and the 2-form C is easily seen to be the Gel'fand–Fuks co-cycle [20]:

$$C(\xi, \eta) := \langle \Sigma(\xi), \eta \rangle = \int_{S^1} \xi(\sigma)''' \eta(\sigma) d\sigma = - \int_{S^1} \xi(\sigma)' d\eta(\sigma)' . \tag{54}$$

Now let us calculate the co-adjoint action of V on V^* :

$$\langle \text{ad}_\xi^* p, \eta \rangle := \int_{S^1} p[\eta, \xi] = \int_{S^1} (2\xi(\sigma)' p(\sigma) + \xi(\sigma) p(\sigma)') \eta(\sigma) d\sigma \tag{55}$$

and as this is satisfied for any $\eta \in V$, we have

$$\text{ad}_\xi^* p = (2\xi(\sigma)' p(\sigma) + \xi(\sigma) p(\sigma)') d\sigma \otimes d\sigma . \tag{56}$$

In particular for any $p \in V^*$

$$\text{ad}_p^* p = 3p(\sigma) p(\sigma)' d\sigma \otimes d\sigma \neq 0 (!!!) . \tag{57}$$

This is a consequence of K being non-invariant.

The equation of motion for the right chiral momentum J^r is again

$$\frac{d}{dt} J^r = -\frac{1}{2} \Sigma(\hat{J}^r) - \frac{1}{2} \text{ad}_{\hat{J}^r}^* J^r \tag{58}$$

but this time the last term does not vanish (compare (31)). In terms of functions this reads

$$\hat{J}^r + \frac{1}{2} (3J^r(J^r)' + (J^r)''') = 0 \tag{59}$$

which is nothing but the equation of Korteweg–de Vries.

Similarly for the left momentum we get

$$\frac{d}{dt} J^l = \frac{1}{2} \Sigma(\hat{J}^l) + \frac{1}{2} \text{ad}_{\hat{J}^l}^* J^l \tag{60}$$

or in the coordinates,

$$j^l - \frac{1}{2} (3J^l(J^l)' + (J^l)''') = 0. \tag{61}$$

Comparing (59) and (61) with the case of loops (45) we see that the chiral equation has been replaced by higher equations of KdV hierarchy. The equations are not linear any longer, but the nonlinearity is in a sense ‘minimal’, as the equations of KdV are completely solvable. The nonlinear terms have their origin precisely in the non-invariance of the Hamiltonian.

5. Concluding remarks

In order to summarize the presented formulation and the examples we shall make some remarks about the underlying geometry.

Let us go back to the Abelian case considered in section 2. The crucial object there is the matrix $L + R$. This matrix is in general non-invertible (as we have seen in section 3, it is generically antisymmetric) and therefore the configuration space splits into its null subspace and the subspace M_{invert} on which $L + R$ is invertible. From (7) and (8) one immediately sees that knowing J^r and J^l it is possible to recover the information about position up to an arbitrary translation along the null space.

On the other hand, from the time evolution of chiral momenta (14), (15) it is clear that it is precisely the restriction to T^*M_{invert} where we observe oscillations, while the motion in the complementary space is free. Therefore we can effectively describe the phase space by the chiral momenta plus initial position and velocity in the null sector. In three dimensions it has a nice interpretation as decomposition into left and right circular polarization.

Let us now discuss the general case. As we have already noticed, the Hamiltonian is a pull-back of a function on $\mathcal{G}^* \times \mathcal{G}^*$ by the chiral momenta (22), (23). This means that one can try to describe the motion in terms of \mathcal{G}^* -valued momenta as their equations of motion are much simpler. Thus one should consider the image of T^*G in $\mathcal{G}^* \times \mathcal{G}^*$ under the map

$$J := J^r \times -J^l. \tag{62}$$

From the definitions (22), (23) it follows that the chiral momenta are not independent, or to be more precise,

$$-J^l(g, p) = \text{Ad}_g^*(J^r(g, p)) + \theta^+(g) \tag{63}$$

i.e. they are on the same orbit of the affine action defined by the co-cycle θ^+ :

$$\mathcal{A}^+(g) = \text{Ad}_g^* + \theta^+(g). \tag{64}$$

Therefore the image of T^*G under (62) is precisely the fibred product defined by projection π on the space of affine orbits:

$$J(T^*G) = \mathcal{G}^* \times_{\pi} \mathcal{G}^* \tag{65}$$

where π assigns to each element $\xi \in \mathcal{G}^*$ its conjugacy class with respect to the action \mathcal{A}^+ . It is clear that the point in a fibred product contains less information about the system than a point in T^*G . In order to see what we have lost let us look at the fibre of J over the point $(\xi, \xi') \in \mathcal{G}^* \times_{\pi} \mathcal{G}^*$. The fibre is clearly isomorphic to the affine stabilizer subgroup of ξ (or any other point conjugated with ξ by \mathcal{A}^+).

Let \mathcal{W} be the set of conjugacy classes (affine orbits). Defining the projection $\pi_J := \pi \circ J$ of T^*G on \mathcal{W} one can see that for any point $r \in \mathcal{W}$

$$\pi_J^{-1}(r) \cong H \times G/H \times G/H \tag{66}$$

where H is the affine stabilizer of some η in class r . Obviously H as a subgroup depends on the choice of η (r fixes it up to an isomorphism only). One can thus immediately see the possible obstructions for the fibres to fit together into the structure of a smooth bundle over \mathcal{W} . However, the situation is not hopeless because in many interesting cases the mapping π defines a trivial fibration over the open subset \mathcal{W}_o of \mathcal{W} . The inverse image $\pi_J^{-1}(\mathcal{W}_o) =: (T^*G)_o$ is called the set of regular points.

In the case of compact Lie groups or loop group the space \mathcal{W}_o can be identified with the interior of the Weyl Chamber in \mathfrak{t}^* , where \mathfrak{t} is the Lie algebra of some maximal torus of G . Then the layers of (66) fit together to give the principal bundle structure

$$H \hookrightarrow (T^*G)_o \xrightarrow{J} (\mathcal{G}^* \times_{\pi} \mathcal{G}^* \simeq G/H \times G/H) \xrightarrow{\pi} \mathcal{W}_o. \tag{67}$$

This is not the end of the story, because the above structure factorizes. Let us consider two copies $P_{l,r}$ of the manifold $\mathcal{W}_o \times G$. endowed with the following symplectic structures:

$$\Omega_l = d\langle r_l, g_l^{-1} dg_l \rangle + \frac{1}{2} \langle \Sigma^+(g_l^{-1} dg_l), g_l^{-1} dg_l \rangle \tag{68}$$

$$\Omega_r = d\langle r_r, dg_r g_r^{-1} \rangle + \frac{1}{2} \langle \Sigma^+(dg_r g_r^{-1}), dg_r g_r^{-1} \rangle \tag{69}$$

and let $P = P_l \times P_r$ be a symplectic manifold with the form $\hat{\Omega} = \Omega_l - \Omega_r$. Then

$$(T^*G_o, \Omega) \simeq (P, \hat{\Omega}) / (r_l - r_r = 0) \tag{70}$$

where $/$ denotes the symplectic reduction by the set of (first-class) constraints. We call the above statement the chiral splitting because $P_{l,r}$ describe the dynamics of chiral momenta. The elements $g_{l,r}$ correspond to vertex operators of WZW.

In the particular case of $\mathcal{L}G$ the form Ω_l has the following structure:

$$\Omega_l = d\langle r_l, g_l^{-1} dg_l \rangle + \frac{1}{2} (k^l + k^r) \langle (u^{-1} du)', u^{-1} du \rangle + d\langle r_l, \text{Ad}_{g_l}^{-1} u^{-1} du \rangle \tag{71}$$

where g_l is a constant loop (the zero mode) and u is an element of the group of loops based at the unity. The second term in Ω_l describes the canonical symplectic structure on the manifold of based loops. It is known that this manifold admits a complex structure which is compatible with the symplectic one [12] and therefore it defines the Kaehler structure. This property is very important for quantization. From (71) one can see that kinematically the chiral sector splits into so-called zero-modes and higher, oscillating modes, described by based loops. The zero-mode dynamics is nothing but a chiral part of free motion of a 'point particle' on a group manifold [9]. The third term of (71) couples the zero-modes to the oscillating ones. Its presence forces the so-called screening, i.e. it sets the upper limit for values of the variable r_l in terms of $k_l + k_r$. It is only with this restriction that the symplectic form is non-degenerate. Because r_l labels the representations of the current algebra the above condition restricts the 'spin' content of the theory. The parametrization of the chiral sectors used above is slightly different from those used in [3] and [6]. It is the initial conditions that are used to parametrize the phase space rather than 'right-' and 'left-movers'. Of course, there is a relation between both descriptions, but the detailed presentation of it is far beyond the scope of this paper. The only comment we can make here is that the parametrization used in this paper corresponds to the restriction of monodromy of the chiral solution to the maximal torus. The restricted chiral sectors are sufficient to describe a general solution of WZW.

It would be interesting to perform a similar analysis for the case of $\text{Diff}_0 S^1$. In this case the structure of the orbit space is a little bit more complicated [16, 17], but still controllable. This gives one some hope that the canonical quantization of the corresponding theory can be performed. We shall return to this issue in a future paper, as well as to the question of the canonical geometry of groups of maps.

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